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Generators of Certain Radical Algebras

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Let C be the Banach algebra of functions continuous in [0, 1], with $\|f\|_C = \max_{0\leqslant x\leqslant 1} |f(x)|$, and multiplication defined by

$$(f * g)(x) = \int_0^x f(t)\, g(x-t)\, dt.$$

For any $f \in C$, $f^{*n}(0) = 0$, $n = 1, 2, 3, \ldots$, so that it is clear that a necessary condition for $f \in C$ to generate C is that $f(0) \neq 0$, but we do not know if this condition is sufficient.

However, let T be the Banach algebra of functions $f(x) = \sum_{n=0}^{\infty} a_n x^n$, with $\|f\|_T = \sum_{n=0}^{\infty} |a_n| < \infty$ and multiplication defined as above. We shall show in Theorem 2 that the obvious necessary condition for $f \in T$ to generate T, i.e., that $f(0) \neq 0$, is also sufficient.

First, though, let us consider another case. Let C_0 be the subalgebra of C consisting of the functions in C which vanish at zero. Clearly, a necessary condition for $f \in C_0$ to generate C_0 is that f does not vanish throughout any interval $[0, a]$, $a > 0$. However, this condition is not sufficient.

Consider

$$h(x) = \frac{e^{-1/x}}{\sqrt{\pi}\, x^{3/2}}, \quad 0 < x \leqslant 1; \quad h(0) = 0.$$

This h belongs to C_0 and is positive except at zero. Let

$$h_1(x) = \frac{e^{-1/x}}{\sqrt{\pi}\, x^{3/2}}, \quad x \in (0, \infty); \quad h_1(0) = 0.$$

We consider the Laplace tranform $L(f) = \int_0^\infty f(x)\, e^{-sx}\, dx$, and recall that $L(f * g) = L(f)\, L(g)$.

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It is a well-known theorem of Boole that for any integrable function F, $\int_{-\infty}^{\infty} F(x)\,dx = \int_{-\infty}^{\infty} F(x - 1/x)\,dx$. We then have

$$\begin{aligned} L(h_1) = \int_0^\infty \frac{e^{-1/x}e^{-sx}}{\sqrt{\pi}\,x^{3/2}}\,dx &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \exp\left[-\left(t^2 + \frac{s}{t^2}\right)\right] dt, \qquad \left(t^2 = \frac{1}{x}\right), \\ &= \frac{s^{1/4}}{\sqrt{\pi}}\int_{-\infty}^{\infty} \exp\left[-\sqrt{s}\left(u^2 + \frac{1}{u^2}\right)\right] du, \ \left(u^2 = \frac{t^2}{\sqrt{s}}\right), \\ &= \frac{e^{-2\sqrt{s}}s^{1/4}}{\sqrt{\pi}}\int_{-\infty}^{\infty} \exp\left[-\sqrt{s}\left(u - \frac{1}{u}\right)^2\right] du, \\ &= \frac{e^{-2\sqrt{s}}s^{1/4}}{\sqrt{\pi}}\int_{-\infty}^{\infty} \exp[-\sqrt{s}\,u^2]\,du, \\ &= \frac{e^{-2\sqrt{s}}}{\sqrt{\pi}}\int_{-\infty}^{\infty} \exp[-v^2]\,dv, \qquad (v^2 = \sqrt{s}\,u^2) \\ &= e^{-2\sqrt{s}}, \end{aligned}$$

so that $L(h_1^{*n}) = e^{-2n\sqrt{s}}$, $n = 1, 2, 3, \ldots$. But

$$\int_0^\infty \frac{ne^{-n^2/x}e^{-sx}}{\sqrt{\pi}\,x^{3/2}}\,dx = \int_0^\infty \frac{e^{-1/x}e^{-sn^2x}}{\sqrt{\pi}\,x^{3/2}}\,dx = e^{-2\sqrt{sn^2}} = e^{-2n\sqrt{s}},$$

so that

$$h_1^{*n}(x) = \frac{ne^{-n^2/x}}{\sqrt{\pi}\,x^{3/2}},$$

and thus

$$h^{*n}(x) = \frac{ne^{-n^2/x}}{\sqrt{\pi}\,x^{3/2}}, \qquad 0 < x \leqslant 1; \qquad h^{*n}(0) = 0.$$

The space C_0, considered as a subset of $L^2[0, 1]$, is dense in $L^2[0, 1]$ and $\| f \|_{C_0} \geqslant \| f \|_{L^2}$. By an application of Müntz' theorem [1, p. 43], $\{(e^{-n^2/x})/(x^{3/2})\}_{n=1}^{\infty}$ is not complete in $L^2[0, 1]$; thus, it is certainly not complete in C_0, and therefore h does not generate C_0.

We shall now show that $f \in T$ generates T if and only if $f(0) \neq 0$. To do this we must first consider another class of Banach algebras.

Given a sequence $\lambda = \{\lambda_n\}$, $\lambda_n > 0$, for all n, $\lambda_1 = 1$, $\lambda_{n+1}/\lambda_n = \delta_n < 1/n^\epsilon$, $\epsilon > 0$, δ_n monotone decreasing, let R_λ be the algebra of power series $W(z) = \sum_{n=1}^{\infty} a_n z^n$, with $\| W \|_\lambda = \sum_{n=1}^{\infty} \lambda_n \mid a_n \mid < \infty$. We note that $\lambda_n = \prod_{i=1}^{n-1} \delta_i$. Also, R_λ is a radical algebra, so that all elements of R_λ have zero as the only point of their spectrum. Our main theorem will be that $W(z) = \sum_{n=1}^{\infty} a_n z^n \in R_\lambda$ generates R_λ if and only if $a_1 \neq 0$.

LEMMA 1.

$$\frac{\lambda_{\alpha_1 t_1 + \cdots + \alpha_s t_s + n - j}}{\prod_{k=1}^{s} (\lambda_{t_k})^{\alpha_k}} \leqslant \frac{\lambda_{n+(l-1)j}}{(\lambda_l)^j},$$

where

$$t_i \geqslant l, \qquad i = 1, 2, \ldots, s, \qquad \sum_{l=1}^{s} \alpha_i = j,$$

and where all of t_i, l, α_i, n, j *are positive integers.*

Proof. If $t_i = l$, $i = 1, 2, \ldots, s$, then we have equality. If we increase t_i by 1, then the ratio of the new left-hand side to the original one is

$$\left(\frac{\lambda_{\alpha_1 t_1 + \cdots + \alpha_s t_s + \alpha_i + n - j}}{\lambda_{\alpha_i, t_1 + \cdots + \alpha_s t_s + n - j}}\right)\left(\frac{\lambda_{t_i}}{\lambda_{t_i+1}}\right)^{\alpha_i},$$

$$= \frac{\delta_{\alpha_1 t_1 + \cdots + \alpha_s t_s + n - j} \cdots \delta_{\alpha_1 t_1 + \cdots + \alpha_s t_s + \alpha_i + n - j}}{(\delta_{t_i})^{\alpha_i}} \leqslant 1,$$

because δ_n is monotone decreasing. The assertion follows by induction.

LEMMA 2. *If* $V(z) = \sum_{m=l}^{\infty} a_m z^m$, $\| V \|_\lambda = K$, *then*

$$\| z^{n-j} V^j \|_\lambda \leqslant \frac{K^j \lambda_{n+(l-1)j}}{(\lambda_l)^j}.$$

Proof. We can write $V(z) = \sum_{m=l}^{\infty} b_m f_m(z)$, where $b_m = a_m \lambda_m / K$ and $f_m(z) = K z^m / \lambda_m$ so that $\sum_{m=l}^{\infty} | b_m | = 1$ and $\| f_m \|_\lambda = K$, $m = l, l+1, \ldots$
Then,

$$z^{n-j} V^j = z^{n-j} \sum_{\alpha_1 + \cdots + \alpha_s = j} \frac{j!}{\alpha_1! \cdots \alpha_s!} b_{t_1}^{\alpha_1} \cdots b_{t_s}^{\alpha_s} f_{t_1}^{\alpha_1} \cdots f_{t_s}^{\alpha_s}$$

$$= z^{n-j} \sum_{\alpha_1 + \cdots + \alpha_s = j} \frac{j!}{\alpha_1! \cdots \alpha_s!} b_{t_1}^{\alpha_1} \cdots b_{t_s}^{\alpha_s} K^j \frac{z^{\alpha_1 t_1 + \cdots + \alpha_s t_s}}{(\lambda_{t_1})^{\alpha_1} \cdots (\lambda_{t_s})^{\alpha_s}},$$

and

$$\| z^{n-j} V^j \|_\lambda \leqslant \sum_{\alpha_1 + \cdots + \alpha_s = j} \frac{j!}{\alpha_1! \cdots \alpha_s!} | b_{t_1} |^{\alpha_1} \cdots | b_{t_s} |^{\alpha_s} K^j \frac{\lambda_{\alpha_1 t_1 + \cdots + \alpha_s t_s + n - j}}{(\lambda_{t_1})^{\alpha_1} \cdots (\lambda_{t_s})^{\alpha_s}}$$

$$\leqslant \sum_{\alpha_1 + \cdots + \alpha_s = j} \frac{j!}{\alpha_1! \cdots \alpha_s!} | b_{t_1} |^{\alpha_1} \cdots | b_{t_s} |^{\alpha_s} K^j \frac{\lambda_{n+(l-1)j}}{(\lambda_l)^j} \quad \text{(by Lemma 1).}$$

$$= \frac{K^j \lambda_{n+(l-1)j}}{(\lambda_l)^j}$$

LEMMA 3. *If* $W(z) = z + \sum_{m=l}^{\infty} a_m z^m$, $l > 1/\epsilon + 1$, *then* $\| W^n \|_\lambda < \lambda_n e^{\|W\|_\lambda - 1}$ *for* $n > (\lambda_l)^{-1/(l-1)\epsilon - 1}$.

Proof. Let $V(z) = \sum_{n=l}^{\infty} a_n z^n$, $\| V \|_\lambda = K$. Then,

$$\| W^n \|_\lambda = \|(z + V)^n \|_\lambda = \left\| \sum_{j=0}^{n} \binom{n}{j} z^{n-j} V^n \right\|_\lambda$$

$$\leqslant \sum_{j=0}^{n} \binom{n}{j} \| z^{n-j} V^j \|_\lambda$$

$$\leqslant \sum_{j=1}^{n} \binom{n}{j} \frac{K^j \lambda_{n+(l-1)j}}{(\lambda_l)^j} \qquad \text{(by Lemma 2)}$$

$$= \lambda_n \sum_{j=0}^{n} \frac{n!}{(n-j)!} \frac{\lambda_{n+(l-1)j}}{(\lambda_l)^j \lambda_n} \frac{K^j}{j!}$$

$$= \lambda_n \sum_{j=0}^{n} \frac{n!}{(n-j)!} \frac{\delta_n \cdots \delta_{n+(l-1)j-1}}{(\lambda_l)^j} \frac{K^j}{j!}$$

$$< \lambda_n \sum_{j=0}^{n} \frac{n!}{(n-j)!} \frac{(\delta_n)^{(l-1)j}}{(\lambda_l)^j} \frac{K^j}{j!}$$

$$< \lambda_n \sum_{j=0}^{n} \frac{n^j}{n^{(l-1)\epsilon j} (\lambda_l)^j} \frac{K^j}{j!}$$

$$= \lambda_n \sum_{j=0}^{n} \left(\frac{n^{1-(l-1)\epsilon}}{\lambda_l} \right)^j \frac{K^j}{j!}$$

$$< \lambda_n \sum_{j=0}^{n} \frac{K^j}{j!} \qquad (\text{for } n > (\lambda_l^{-1/(l-1)\epsilon - 1})$$

$$< \lambda_n e^K = \lambda_n e^{\|W\|_\lambda - 1}.$$

LEMMA 4. *If* $W(z) = z + \sum_{n=l}^{\infty} a_n z^n$ *is in* R_λ, $l > 1/\epsilon + 1$, *and* $(W^{-1}(z))^m = \sum_{n=m}^{\infty} b_n^{(m)} z^n$, *where* $W^{-1}(z)$ *is the formal inverse of* $W(z)$, *then* $| b_N^{(m)} | < (CN^{2-(m-1)\epsilon})/\lambda_N$, *for some constant* C.

Proof. Let $W_N(z) = z + \sum_{n=l}^{N} a_n z^n$, $N > m$. Since $W_N'(0) \neq 0$, $(W_N^{-1}(z))^m$

is analytic in a small enough disc about the origin, bounded, say, by C'. If we let $(W_N^{-1}(z))^m = \sum_{n=m}^{\infty} c_n^{(m)} z^n$, then $b_i^{(m)} = c_i^{(m)}$, $i = m, m+1, \ldots, N+m-1$. Thus,

$$b_N^{(m)} = c_N^{(m)} = \frac{1}{2\pi i} \int_{C'} \frac{[W_N^{-1}(z)]^m}{z^{N+1}}\, dz = \frac{1}{2\pi i} \int_{C''} \frac{z^m W_N'(z)\, dz}{W_N^{N+1}(z)}.$$

Since $\sum_{n=l}^{\infty} \lambda_n \mid a_n \mid < \infty$, one has $\mid a_n \mid \leqslant M/\lambda_n$, for some constant M. Now,

$$\mid W_N(z) \mid \geqslant \mid z + \sum_{n=l}^{N} a_n z^n \mid \geqslant \mid z \mid - \sum_{n=l}^{N} \mid a_n \mid \mid z \mid^n \geqslant \mid z \mid - M \sum_{n=l}^{N} \frac{\mid z \mid^n}{\lambda_n},$$

and

$$\mid W_N'(z) \mid = \left| 1 + \sum_{n=l}^{N} n a_n z^{n-1} \right| \leqslant 1 + M \sum_{n=l}^{N} \frac{n \mid z \mid^{n-1}}{\lambda_n}.$$

We replace C'' by the circle Γ, about the origin, of radius

$$r = \left(\frac{\lambda_N}{N^2}\right)^{1/(N-1)},$$

because, as we show below, $r - M \sum_{n=l}^{N} (r^n/\lambda_n) > 0$, so that $W_N(z) \neq 0$ inside and on Γ (except at the origin). We then obtain

$$\mid b_N^{(m)} \mid \leqslant \frac{1}{2\pi} \int_C \frac{\mid z \mid^m \mid W_N'(z) \mid}{\mid W_N(z) \mid^{N+1}}\, dz \leqslant \frac{r^{m+1}\left(1 + M \sum_{n=l}^{N} \frac{n r^{n-1}}{\lambda_n}\right)}{\left(r - M \sum_{n=l}^{N} \frac{r^n}{\lambda_n}\right)^{N+1}}$$

$$= \frac{r^{m-N}\left(1 + M \sum_{n=l}^{N} \frac{n r^{n-1}}{\lambda_n}\right)}{\left(1 - M \sum_{n=l}^{N} \frac{r^{n-1}}{\lambda_n}\right)^{N+1}}.$$

Now,

$$r = \left(\frac{\lambda_N}{N^2}\right)^{1/(N-1)} \leqslant \left(\frac{1}{N^2 (N-1)!^{\epsilon}}\right)^{1/(N-1)}, \quad r^{m-N} = \frac{r^{m-1}}{r^{N-1}} \leqslant \frac{2 e^{\epsilon m} N^{2-(m-1)\epsilon}}{\lambda_N},$$

so that

$$r < 2\left(\frac{e}{N}\right)^{\epsilon}, \quad \text{and} \quad r^{l-1} < \frac{(2e^{\epsilon})^{l-1}}{N}.$$

By examining the ratio of successive summands of $\sum_{n=l}^{N} (r^{n-1}/\lambda_n)$, we see that the summands decrease first and then increase, so that

$$\sum_{n=l}^{N} \frac{r^{n-1}}{\lambda_n} = \sum_{n=l}^{2l-2} \frac{r^{n-1}}{\lambda_n} + \sum_{n=2l-1}^{N} \frac{r^{n-1}}{\lambda_n}$$

$$\leqslant (l-1) \max\left(\frac{r^{l-1}}{\lambda_l}, \frac{r^{2l-3}}{\lambda_{2l-2}}\right) + (N-2l+2) \max\left(\frac{r^{2l-2}}{\lambda_{2l-1}}, \frac{1}{N^2}\right)$$

$$\leqslant (l-1) \max\left[\frac{(2e^{\epsilon})^{l-1}}{N\lambda_l}, \frac{(2e^{\epsilon})^{2l-3}}{N^{2-\epsilon}\lambda_{2l-2}}\right]$$

$$+ (N-2l+2) \max\left[\frac{(2e^{\epsilon})^{2l-2}}{N^2\lambda_{2l-1}}, \frac{1}{N^2}\right]$$

$$\leqslant \frac{C_1}{N}, \qquad \text{where} \qquad C_1 = \frac{(l-1)(2e^{\epsilon})^{2l-3}}{\lambda_{2l-2}} + \frac{(2e^{\epsilon})^{2l-2}}{\lambda_{2l-1}}.$$

Similarly,

$$\sum_{n=l}^{N} \frac{nr^{n-1}}{\lambda_n} \leqslant (N-l+1) \max\left(\frac{lr^{l-1}}{\lambda_l}, \frac{1}{N}\right)$$

$$\leqslant (N-l+1) \max\left[\frac{l(2e^{\epsilon})^{l-1}}{N\lambda_l}, \frac{1}{N}\right]$$

$$\leqslant C_2, \qquad \text{where} \qquad C_2 = \frac{l(2e^{\epsilon})^{l-1}}{\lambda_l}.$$

Hence,

$$b_N^{(m)} \leqslant \frac{2e^{\epsilon m}N^{2-(m-1)\epsilon}(1+MC_2)}{\lambda_N\left(1-\dfrac{MC_1}{N}\right)^{N+1}} < \frac{C_3N^{2-(m-1)\epsilon}}{\lambda_N},$$

where $C_3 = 4e^{\epsilon m}(1+MC_2)\, e^{MC_1}$.

LEMMA 5. *$f(z) = \sum_{n=1}^{N} a_n z^n$ generates R_λ if $a_1 \neq 0$.*

Proof. Let D be the closed sub-algebra generated by f. The spectrum of f consists of the point 0 and f^{-1} is analytic at 0 (because $a_1 \neq 0$). Hence $f^{-1}(f(z))$ belongs to D (2, p. 78]. But $f^{-1}(f(z)) = z$ and z generates R_λ; thus $f(z)$ generates R_λ.

THEOREM 1. *$W(z) = \sum_{n=1}^{\infty} a_n z^n \in R_\lambda$ generates R_λ if and only if $a_1 \neq 0$.*

Proof. If $a_1 = 0$, then all powers of W also have zero as their coefficient of z and W does not generate R_λ.

If $a_1 \neq 0$, we can assume without loss of generality that $a_1 = 1$.

Clearly, $W_1(z) = z + \sum_{n=l}^{\infty} c_n z^n$, $l = [1/\epsilon] + 2$, is generated by $W(z)$, for some $\{c_n\}_{n=l}^{\infty}$. Consider $(W_1^{-1}(z))^m = \sum_{n=m}^{\infty} b_n^{(m)} z^n$, $m \geqslant 4/\epsilon + 1$, and let $f_N(z) = \sum_{n=m}^{N} b_n^{(m)} z^n$. Then

$$\|[W_1^{-1}(W_1(z))]^m - f_N(W_1(z))\|_\lambda$$

$$= \left\| \sum_{n=N+1}^{\infty} b_n^{(m)} W_1{}^n(z) \right\|_\lambda$$

$$\leqslant \sum_{n=N+1}^{\infty} | b_n^{(m)} | \, \| W_1{}^n(z) \|_\lambda$$

$$< C_3 \sum_{n=N+1}^{\infty} \frac{n^{2-(m-1)\epsilon}}{\lambda_n} \lambda_n e^{\|W_1\|_\lambda - 1} \qquad (\text{for } N > (\lambda_l)^{-1/((l-1)\epsilon - 1)})$$

$$\leqslant C_3 e^{\|W_1\|_\lambda - 1} \sum_{n=N+1}^{\infty} \frac{1}{n^2},$$

which can be made arbitrarily small, so that $f_N(W_1(z) \to [W_1^{-1}(W_1(z))]^m = z^m$. Hence, $\{z^m\}_{m=[4/\epsilon]+2}^{\infty}$ is generated by W_1 and therefore, by W. Thus, $W(z) - \sum_{n=[4/\epsilon]+2}^{\infty} a_n z^n = z + \sum_{n=2}^{[4/\epsilon]+1} a_n z^n$ is generated by W. But $z + \sum_{n=2}^{[4/\epsilon]+1} a_n z^n$ generates R_λ, by Lemma 5, so that W does the same. This completes the proof.

Let R be the algebra obtained by setting $\lambda_n = 1/(n-1)!$

THEOREM 2. *$f(x) = \sum_{n=0}^{\infty} a_n x^n \in T$ generates T if and only if $f(0) \neq 0$.*

Proof. If $f(x) = \sum_{n=0}^{\infty} a_n x^n \in T$, then $\sum_{n=0}^{\infty} n!\, a_n z^{n+1} \in R$. Consider the transformation L from T into R, given by $L(\sum_{n=0}^{\infty} a_n x^n) = \sum_{n=0}^{\infty} n!\, a_n z^{n+1}$. We have $L(x^n) = n!\, z^{n+1} = \int_0^{\infty} x^n e^{-x/z}\, dx$, so that $L(x^n)$ equals the Laplace transform of x^n, $0 \leqslant x < \infty$. Hence, if P and Q are polynomials, $L(P * Q) = L(P)\, L(Q)$ and, since, $L(\sum_{n=0}^{\infty} a_n x^n) = \lim_{N\to\infty} L(\sum_{n=0}^{N} a_n x^n)$ (in R), it follows that $L(f * g) = L(f)\, L(g)$, for all $f, g \in T$. Furthermore, $\| L(\sum_{n=0}^{\infty} a_n x^n)\|_R = \sum_{n=0}^{\infty} | a_n | = \| \sum_{n=0}^{\infty} a_n x^n \|_T$, so that L is an isometric isomorphism from T onto R. By Theorem 1, $L(f) = \sum_{n=1}^{\infty} a_{n-1}(n-1)!\, z^n$ generates R if and only if $a_0 \neq 0$, so that f generates T if and only if $a_0 \neq 0$, i.e., if and only if $f(0) \neq 0$.

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